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Using computer algebra tools to classify serial cuspidal manipulators

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Abstract: In this paper we present a classification of 3-revolute-jointed manipulators based on the cuspidal behaviour. It was shown in a previous work that this ability to change posture without meeting a singularity is equivalent to the existence of a point in the workspace, such that a polynomial of degree four depending on the parameters of the manipulator and on the cartesian coordinates of the effector has a triple root.

More precisely, from a partition of the parameters' space, such that in any connected component of this partition the number of triple roots is constant, we need to compute one sample point by cell, in order to have a full description, in terms of cuspidality, of the different possible configurations.

Key-words: computer algebra, Gröbner basis, cylindrical algebraic decomposition, triangular sets, serial manipulators, semialgebraic sets.

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Utilisation de méthodes formelles pour classifier des robots sériels cuspidaux

Résumé : Dans ce rapport, nous présentons une classification d'une famille de robots sériels à 3 degrés de liberté selon leur comportement cuspidal. Il à été montré, dans des travaux précédents, que tester le caractère cuspidal d'un robot de ce type revient exactement à décider si un polynôme de degré quatre, dépendant des paramètres de conception, admet ou non une racine réelle triple.

Nous calculons une partition de l'espace des paramètres telle que au dessus de chaque composante connexe de cette partition, le nombre de points triples du polynôme soit constant, produisant ainsi une description complète des configurations cuspidales.

Mots-clés : calcul formel, base de Gröbner, décomposition cylindrique algébrique, ensembles triangulaires, robots sériels, ensembles semi-algébriques

1 Introduction

Industrial robotic 3-DOF manipulators are currently designed with very simple geometric rules on the designed parameters, the ratios between them are always of the same kind. In order to enlarge the possibilities of such manipulators, it may be interesting to relax the constraints on the parameters.

The behavior of the manipulators when changing posture depends strongly on the design parameters and it can be very different from the one of manipulators commonly used in Industry.

P. Wenger and J. El Omri [6], [11] have shown that for some choices of the parameters, 3-DOF manipulators may be able to change posture without meeting a singularity in the joint space. This kind of manipulators is called **cuspidal**. It is worth noting that in case of obstructed environment, this property would yield more flexibility which can be very useful in practice for industrial purpose. They succeed in characterizing 3-revolute jointed manipulators using a homotopy based classification scheme [10], but they needed general conditions on the design parameters, more precisely they wanted to find answers to the following issues:

- **Problem 1** : For given parameters, is the manipulator cuspidal?
- **Problem 2** : How to obtain an algebraic set in the parameters' space such that in each connected components of the complementary, the cuspidality behavior is fixed? And then, can we caracterize each connected component with a test manipulator?

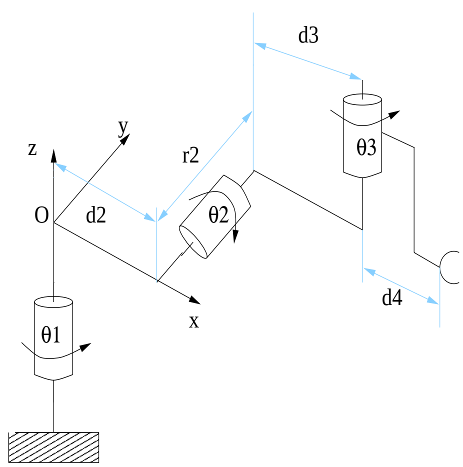

Figure 1: The manipulator under our hypothesis

We restrict the study to 3-DOF manipulators as described in figure 1. The geometrical modelisation of the manipulator is explained in the first section of this paper. As recalled in the second section, testing if such a manipulator is cuspidal or not is equivalent to deciding if an algebraic set has real roots or not, this set corresponding to the existence of a triple root of a given polynomial of the fourth degree.

We manage to answer both questions under a few hypothesis which are not restricting ones according to roboticians : for example it is impossible to construct, in practice, a manipulator whose parameters lies on a strict hypersurface of the space of the parameters.

We explain in the third section how we obtained the algebraic set, depending only on the parameters, which complementary interests us for the study of the cuspidality behavior. Then in the fourth section, we show how to compute a partition of the space of parameters, such that in each cell of maximal dimension, the behavior of the manipulator is known (cuspidal or not), while the other cells are embedded inside strict algebraic subsetsof the parameters' space.

For the computations, we use very recent tools from computationnal algebra, as for examples, the algorithms of F.Rouillier [7] [8] for solving zero-dimensionnel systems,the algorithm $F4$ of J.C.Faugère [5] for computing Gröbner Bases, the works of P.Aubry on the trangular sets [1] [2] and the implementation of the Descartes' rule of sign based algorithm done by F.Rouillier and P.Zimmerman [9].

2 The kinematic map

The 4 design parameters are d_2, d_3, d_4 and r_2. In order to normalize (d_2, d_3, d_4, r_2) and to reduce the number of parameters, we assume that d_2 is equal to 1.

So the space of design parameters is $(\mathbb{R}^+)^3$. Along our study of the behavior of the manipulator, we will work into two spaces:

- the **joint space** described by the joint variables $(\theta_1, \theta_2, \theta_3)$, this space is isomorphic to $]-\pi,\pi]^3$ as the joint limits will be ignored. Moreover, as the manipulator turns around the axe of the first revolute joint, the joint space can be represented by the torus (in the variables (θ_2, θ_3).

- the **task space** representing the position of the end-effector in the Cartesian coordinates (x, y, z), which is isomorphic to $\mathbb{R}^3$ if we suppose there are no obstacles.

The kinematic map f maps the joint space on the task space :

$$\begin{array}{rcccl} f & : &]-\pi,\pi]^3 & \longrightarrow & \mathbb{R}^3 \\ & & (\theta_1,\theta_2,\theta_3) & \longmapsto & (x,y,z) \end{array}$$

The image of this map in the task space is called the **workspace**.

To express f, we just have to express the changement of basis between the basis of the origin and the basi of the effector, which gives us the following expression for f:

$$\left\{ \begin{array}{lcl} x & = & (d_3+\cos\theta_3 d_4)(\cos\theta_1\cos\theta_2) + (r_2+d_4\sin\theta_3)\sin\theta_1 + \cos\theta_1 \\ y & = & (d_3+\cos\theta_3 d_4)(\sin\theta_1\cos\theta_2) - (r_2+d_4\sin\theta_3)\cos\theta_1 + \sin\theta_1 \\ z & = & (d_3+\cos\theta_3 d_4)\sin\theta_2 \end{array} \right. \quad (1)$$

For the user, it is of first importance to know the **workspace**, he wants to be able to divide the task space into accessible or not accessible zones, and to know the number of joint's configurations corresponding to the same position.

3 Algebraic characterization of cuspidal manipulators

In the classical cases used in Industry, manipulators, to change posture, need to pass through a singularity of the joint space. In other words, the end-effector must bump into the frontiere of the workspace.

But this behavior is not general at all. It was shown by P. Wenger that a 3-DOF manipulator can execute a non singular change of posture if and only if there exist at least one point in its workspace with exactly three coincident inverse kinematic solutions (corresponding in a cross section of the workspace to cusp points, hence the word **cuspidal**).

As it is difficult to express such a condition directly from the kinematic map, the idea is to eliminate joint variables from the system, let say θ_1 and θ_2, in order to obain a condition over the last joint variable.

By taking $s_i = sin(\theta_i)$ and $c_i = cos(\theta_i), i = 1 \ldots 3$, and adding the algebraic relations $s_i^2 + c_i^2 = 1 \; i = 1 \ldots 3$ to the equations defining f, we have to study an algebraic system of equations.

In order to eliminate θ_1 and θ_2, we compute a Gröbner Basis of the system under the following elimnatiing order:

$$[c_1, s_1, c_2, s_2] > [c_3, s_3, r_2, d_3, d_4, x, y, z].$$

The Gröbner basis G_3 of the Elimination Ideal is composed of two polynomials :

$$c_3^2 + s_3^2 - 1$$

and

$$m_5 c_3^2 + m_4 s_3^2 + m_3 c_3 s_3 + m_2 c_3 + m_1 s_3 + m_0$$

where :

$$\left\{ \begin{array}{lcl} m_0 & = & -x^2 - y^2 + r_2^2 + \frac{(R+1-L)^2}{4} \\ m_1 & = & 2r_2 d_4 + (L - R - 1) d_4 r_2 \\ m_2 & = & (L - R - 1) d_4 d_3 \\ m_3 & = & 2 r_2 d_3 d_4^2 \\ m_4 & = & d_4^2 (r_2^2 + 1) \\ m_5 & = & d_4^2 d_3^2 \end{array} \right.$$

with $R = x^2 + y^2 + z^2$ and $L = d_4^2 + d_3^2 + r_2^2$.

After a change of variables in $t = \tan(\frac{\theta_3}{2})$, the second polynomial becomes:

$$P(t) = at^4 + bt^3 + ct^2 + dt + e$$

with:

$$\left\{\begin{array}{lcl} a & = & m_5 - m_2 + m_0 \\ b & = & -2m_3 + 2m_1 \\ c & = & -2m_5 + 4m_4 + 2m_0 \\ d & = & 2m_3 + 2m_1 \\ e & = & m_5 + m_2 + m_0 \end{array}\right.$$

Deciding if a manipulator is cuspidal is equivalent to deciding if this polynomial P of degree 4 in t (whose coefficients are polynomial with respect to $x, y, z, d_4, d_3, r2$) admits real triple roots.

It is important to note that every solution in $t = \tan(\frac{\theta_3}{2})$ is uniquely lifted in a 3-uplet $(\theta_1, \theta_2, \theta_3)$, except if $z = 0$ or $x^2 + y^2 = 0$, but this case will be treated later. We will show that the design parameters d_3, d_4, r_2 of manipulators such that $z(x^2 + y^2) = 0$ are in a strict hypersurface of $\mathbb{R}^3$. So, finding parameters' values defining cuspidal manipulators remains to find the values of d_4, d_3, r_2 such that $P(t)$ has triple points by solving the following system (as the boundaries of the workspace form a revolution surface around the axis Oz, it is generically zero-dimensional once the parameters are fixed) :

$$\left\{\begin{array}{lcl} P & = & 0 \\ \frac{\partial P}{\partial t} & = & 0 \\ \frac{\partial^2 P}{\partial t^2} & = & 0 \end{array}\right. \tag{2}$$

4 The algebraic set defining the classification

We want to obtain a partition of the space of parameters, such that the cuspidality behavior of the manipulators corresponding to the same cell is the same.

As it is impossible to construct in practice a manipulator whose parameters correspond exactly to given parameters, we just deal with "generic" solutions. In other words, we are only interested in the cells of maximal dimension. In each of those cells we want that the number of real roots of the system is constant (in the generic cases it appears that the dimension of thr system when we fix the parameters is zero).

In order to obtain some conditions on the parameters, we need to eliminate the variables t, Z, R from the system of equations. The first idea would be to compute a Gröbner Basis of the system with respect to an eliminating order. This is possible using the most recents algorithms from J.C.Faugère ([5]). But the result obtained is very huge and difficult to use because all the multiplicities as well as non real components of the variety, are kept along the computation.

We need to obtain an eliminating polynomial depending on the parameters d_4, d_3, r_2 and on one of the three variables t, R and Z, with uniquely defined solutions with respect to the two remaining variables in the fibers.

Theoretically, the possibility of getting such an equivalent system without loosing the algebraic structure (ideal) depends strongly on the shape of the solutions and it is in general not possible.

So, we choose to represent the solutions of system 2 as *regular* zeroes of triangular set (with respect to the terminology of [2]), with the hope that the non regular solutions correspond to values of the parameters contained in strict hypersurfaces of the space of the parameters.

As the direct computation do not give a pleasant result, we decided to modify the variables and to simplify the problem. Thanks to the following change of variables:

$$\left\{\begin{array}{lclcl} U & = & e-a & = & 2(d_4^2+d_3^2+r_2^2-R-1)d_4r_2 \\ X_1 & = & d-b=d-X_3-hU & = & 8r_2d_3d_4^2 \\ X_2 & = & U+2a-c & = & 4d_4^2(d_3^2-r_2^2-1) \\ h & = & r_2/d_3 & & \\ X_3 & = & b-hU & = & 4r_2d_4(1-d_3d_4) \end{array}\right.$$

the Gröbner basis of the new system with respect to the lexicographic order :

$$t>a>U>X_1>X_2>X_3>h$$

is so easy to compute that it can be done vey quickly with the algorithm of Buchberger implemented under MAPLE.

The basis obtain as the following form :

$$\left\{\begin{array}{l} surf_U(U,X_1,X_2,X_3,h)=0 \\ g_{a,1}(a,U,X_1,X_2,X_3,h)=0 \\ g_{a,2}(a,U,X_1,X_2,X_3,h)=0 \\ \vdots \\ g_{a,14}(a,U,X_1,X_2,X_3,h)=0 \\ g_{t,1}(t,a,U,X_1,X_2,X_3,h)=0 \\ g_{t,2}(t,a,U,X_1,X_2,X_3,h)=0 \\ \vdots \\ g_{t,18}(t,a,U,X_1,X_2,X_3,h)=0 \end{array}\right.$$

where :

- the polynomial $g_{a,1}$ is of degree 1 in the variable a.
- the polynomial $g_{t,1}$ is of degree 1 in the variable t.

We extract from the system the following triangular set:

$$\left\{\begin{array}{l} surf_U(U,X_1,X_2,X_3,h)=0 \\ g_{a,1}=lc_a(X_1,X_2,X_3,h)a+c_a(U,X_1,X_2,X_3,h)=0 \\ g_{t,1}=lc_t(X_1,X_2,X_3,h)t+c_t(U,X_1,X_2,X_3,h)=0 \end{array}\right. \quad (3)$$

The solutions of the system are contained in those three equations except if one of them is identically vanishing.

So coming back to our first variables, we can define the generic solutions of the problem as regular roots of a triangular system with the following shape:

$$\left\{ \begin{array}{l} surf(R, d_4, d_3, r_2) = 0 \\ lc_Z(d_4, d_3, r_2)Z + tr_Z(R, d_4, d_3, r_2) \\ lc_t(d_4, d_3, r_2)t + tr_t(R, Z, d_4, d_3, r_2) \end{array} \right. \tag{4}$$

Now to obtain a description of the complementary of our algebraic set, we can compute directly a Cylindrical Algebraic Decomposition adapted to this set and then just keep the cells of maximal dimension. But the computation of the discriminant of the polynomial $surf$ would be a limiting step.

Let's study the system more deeply.

The set $V_{lc}(Z,t) = \{(d_3, d_4, r_2) | lc_Z(d_3, d_4, r_2) lc_t(d_3, d_4, r_2) = 0\}$ defines an hypersurface in the parameters' space, which is a strict closed set.We can remark that $lc_Z = 0$ has no real roots, so $V_{lc} = \{(d_3, d_4, r_2) |\ lc_t(d_3, d_4, r_2) = 0\}$. Studying the generic solutions of the first system is exactly avoiding the parameters on this V_{lc} and then parameters for which the number of real roots of $surf(R)$ varies. It means the parameters lying in the hypersurface :

$$\{(d_3, d_4, r_2) | lc_{surf,R} = 0 \text{ or } discrim(surf, R) = 0\}$$

where $lc_{surf,R}$ (resp. $discrim(surf, R)$) is the leading term (resp. discriminant) of $surf$ with respect to R with respect to R.

The real hypersurface $lc_{surf,R} = 0$ is empty and the discriminant factorise in three factors:

- $f_1 = d_3^2 - d_4^2 + r_2^2$
- $f_2 = d_4^2 d_3^6 - d_4^4 d_3^4 + 3 d_4^2 d_3^4 r_2^2 - 2 d_4^2 d_3^4 + 2 d_4^4 d_3^2 - 2 r_2^2 d_3^2 d_4^4 + d_4^2 d_3^2 + 3 d_4^2 d_3^2 r_2^4 - d_3^2 r_2^2 - 2 r_2^2 d_4^4 - d_4^4 r_2^4 - d_4^4 + r_2^6 d_4^2 + d_4^2 r_2^2 + 2 r_2^4 d_4^2$
- $f_3 = 2 r_2^8 d_4^2 d_3^2 + 2 r_2^2 d_4^2 d_3^8 + 6 r_2^4 d_4^2 d_3^6 + 6 r_2^6 d_4^2 d_3^4 + 4 r_2^{10} d_4^4 + 6 r_2^{10} d_3^2 d_4^4 + 15 r_2^8 d_3^4 d_4^4 + 15 r_2^4 d_3^8 d_4^4 + 12 r_2^8 d_3^2 d_4^4 - 4 r_2^4 d_3^4 d_4^4 + 6 r_2^8 d_4^4 - 4 d_3^6 d_4^4 + 6 d_3^8 d_4^4 + r_2^{12} d_4^4 - 4 r_2^2 d_4^2 d_3^6 + 4 r_2^6 d_4^2 d_3^2 + 20 r_2^6 d_3^6 d_4^4 - 8 r_2^4 d_3^6 d_4^4 + 8 r_2^6 d_3^4 d_4^4 + d_4^4 r_2^4 + 2 d_4^2 d_3^2 r_2^4 + 2 d_4^2 d_3^4 r_2^2 + d_4^4 d_3^4 - 12 r_2^2 d_3^8 d_4^4 - 2 r_2^2 d_3^2 d_4^4 + d_3^4 r_2^4 + 4 r_2^6 d_3^2 d_4^4 - 4 r_2^4 d_3^2 d_4^4 + 4 r_2^2 d_3^4 d_4^4 + 4 r_2^2 d_3^6 d_4^4 + 4 r_2^6 d_4^4 + d_3^{12} d_4^4 - 4 d_3^{10} d_4^4 + 6 r_2^2 d_3^1 0 d_4^4$

The real hypersurface $f_3 = 0$ is also empty from the real point of view. Also we must remember that the variables Z and $R - Z$ are strictly positive. Substituting $Z = 0$ or $R - Z = 0$ in the first system permits us to find two polynomial conditions on parameters, let say $Z_0(d_3, d_4, r_2) = 0$ and $RZ_0(d_3, d_4, r_2) = 0$.

The polynomial RZ_0, does not present any real root. The polynomial Z_0 factorises in two polynomials :

- $Z_{01} = d_3^2 r_2^2 + d_3^2 - 2 d_3^3 + d_3^4 - d_4^2 + 2 d_3 d_4^2 - d_4^2 d_3^2$
- $Z_{02} = d_3^2 r_2^2 + d_3^2 + 2 d_3^3 + d_3^4 - d_4^2 - 2 d_3 d_4^2 - d_4^2 d_3^2$

Finally, the algebraic set that will define our partition is :

$$\{Z_{01} = 0, Z_{02} = 0, f_1 = 0, f_2 = 0, lc_t = 0, d4 = 0, d3 = 0, r2 = 0\}.$$

It gives us a partition of the space of parameters, such that in each cell of maximal dimensionwhere the parameters $d4, d3, r2$ are strictly positive, the number of cusp points in a cross section of the workspace is constant.

5 Partition's cells computation

A way to represent such cells is now to compute a CAD (Cylindrical Algebraic Decomposition - see [4]) of $\mathbb{R}^3$ adapted to this set of polynomials.

But it may give us a result very huge and difficult to analyse in practice, and with lots of cells we are not intersted in.

In order to obtain a decomposition easier to manipulate, we use the fact that we are just interested in the "generic"solutions, so in the cells of maximal dimension in our partition of the space of parameters.

So we adapt the Collins'Algorithm to our case.

- Projection phase
  At each step we keep the leading coefficients of the polynomials, the resultants and the discriminants, keeping in mind that the polynomials which have no real roots do not interest us. As we are not interested in the multiplicities of the singularities and intersections between hypersurfaces, we won't compute the subresultants.

- Lifting phase
  As we are just interested in the cells of maximal dimension, we can work only with rationnal test points (there are no computations with real algebraic numbers).

5.1 The projection phase

Let A be the algebraic set $\{f_1, f_2, , lc_t, Z_{01}, Z_{02}\}$. Let

$$PROJ_{d_4}(A) = \{A \cap \mathbb{Q}[d_3, r_2], disc(P, d_4), res(P, Q, d_4), lc(P) \mid P, Q \in A\}.$$

After computations we obtain:

$$PROJ_{d_4}(A) = \{C_1, ..., C_6\}.$$

where:

$$
\left\{
\begin{array}{lll}
C_1 = & d_3 - 1 & = 0, \\
C_2 = & -1 + 2\,d_3 & = 0, \\
C_3 = & -{d_3}^2 + {d_3}^4 + 2\,{d_3}^2{r_2}^2 + {r_2}^2 + {r_2}^4 & = 0, \\
C_4 = & {d_3}^2{r_2}^2 + 2\,{d_3}^2 - 2\,d_3\,{r_2}^2 - 4\,d_3 + 3\,{r_2}^2 + 3 + {r_2}^4 & = 0, \\
C_5 = & -3\,{d_3}^2 + 3\,{r_2}^2 + {d_3}^4{r_2}^2 + 2\,{d_3}^4 + 3\,{d_3}^2{r_2}^4 + 4\,{d_3}^2{r_2}^2 + 3\,{r_2}^6 + 6\,{r_2}^4 & = 0, \\
C_6 = & 3\,{d_3}^2 - 3\,{r_2}^2 + 10\,{d_3}^4 - 10\,{d_3}^2{r_2}^2 - 3\,{r_2}^4 + 10\,d_3\,{r_2}^2 - 10\,{d_3}^3 - 5\,{d_3}^6 - 11\,{d_3}^4{r_2}^2 - & \\
 & 7\,{d_3}^2{r_2}^4 - {r_2}^6 + 8\,{d_3}^3{r_2}^2 + 8\,d_3\,{r_2}^4 + 2\,d_3\,{r_2}^6 + 6\,{d_3}^3{r_2}^4 + 6\,{d_3}^5{r_2}^2 + 2\,{d_3}^7 & = 0,
\end{array}
\right. \tag{5}
$$

We can visualise ... pace (d_3, r_2) it implies, on the follo...

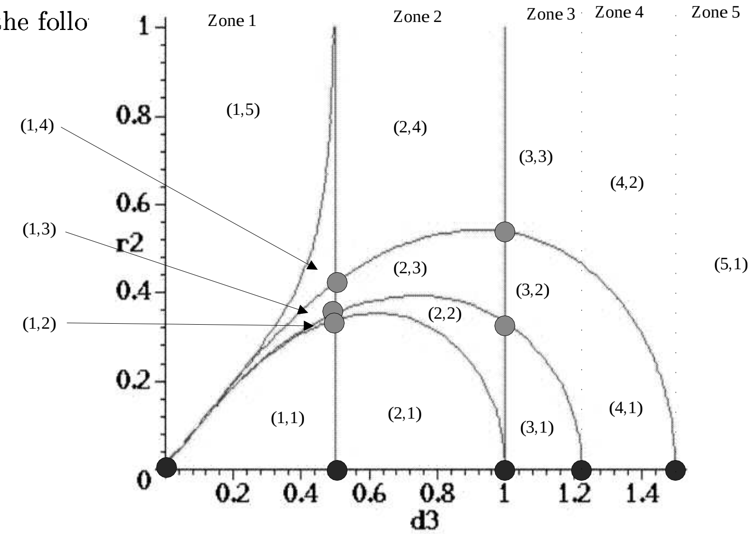

Figure 2: partition of the parameters' space (d_3, r_2)

Then

$$PROJ_{r_2}(PROJ_{d_4}(A)) = \{-1 + 2d_3, d_3 - 1, 2d_3 - 3, -3 + 2d_3^2\}.$$

5.2 The lifting phase

At each step of the lifting phase we use the Descartes' rule of sign based algorithm implemented by F.Rouillier and P.Zimmerman [9]. to find isolating intervals of the real roots appearing along the computation. Then we chose a rationnal point between each interval encontered.

For the first step we obtained the following test values for d_3:

- $d_3^{(1)} = \frac{219}{1024} \in]0, P_2[$,
- $d_3^{(2)} = \frac{3}{4} \in]P_2, P_3[$,
- $d_3^{(3)} = \frac{1139}{1024} \in]P_3, P_4[$,
- $d_3^{(4)} = \frac{2791}{2048} \in]P_4, P_5[$,
- $d_3^{(5)} = \frac{1011}{512} \in]P_5, \infty[$,

After substituting those values in the list $\{C_1, ..., C_6\}$, we obtain 6 polynomials in r_2. In each case we isolate the real roots and obtain the following test values:

- $d_3^{(1)} = \frac{219}{1024} : \{\frac{403}{4096}, \frac{403}{2048}, \frac{827}{4096}, \frac{215}{1024}, \frac{13657}{64000}\}$,
- $d_3^{(2)} = \frac{3}{4} : \{\frac{339}{2048}, \frac{371}{1024}, \frac{939}{2048}, \frac{67189}{128000}\}$,
- $d_3^{(3)} = \frac{1139}{1024} : \{\frac{259}{2048}, \frac{785}{2048}, \frac{32907}{64000}\}$,
- $d_3^{(4)} = \frac{2791}{2048} : \{\frac{181}{1024}, \frac{45439}{128000}\}$,
- $d_3^{(5)} = \frac{1011}{512} : \{1\}$.

Then we do the same in A. Over each 15 cells of the plane (r_2, d_3) we obtain 7 test values in the parameter d_4, so we eventually found 105 test points (each one corresponding to a cell of maximal dimension).

5.3 Results

For each sample point, it is then sufficent to solve the zero-dimensionnal system ([7] [8]) (counting the number of real roots), after specialisation adding the equations $T_1Z^2 - 1 = 0$ and $T_2(R - Z)^2 - 1 = 0$ to discriminate the admissible solutions, to get the numbe of cusp points in a cross section of the workspace corresponding to the selected parameters.

This can be done computing a Rationnal Univariate Representation of the system (([8]) and then isolating the real roots of this RUR with the Descartes' rule of sign based algorithm ([9]).

As $Z = z^2$, the number of cusp points appearing in a cross section of the workspace, equals twice the number of real roots of the RUR.

Here are the results for each sample point:

$(d_3, r_2) \setminus d_4$	1	2	3	4	5	6	7
(1,1)	0	0	4	4	2	0	0
(1,2)	0	4	4	4	2	0	0
(1,3)	0	4	4	4	2	0	0
(1,4)	0	4	4	2	2	0	0
(1,5)	0	4	4	2	0	0	0
(2,1)	0	0	4	4	2	2	0
(2,2)	0	4	4	4	2	2	0
(2,3)	0	4	4	4	2	2	0
(2,4)	0	4	4	2	2	2	0
(3,1)	0	4	4	4	2	2	4
(3,2)	0	4	4	4	2	2	4
(3,3)	0	4	4	2	2	2	4
(4,1)	0	4	4	4	2	2	4
(4,2)	0	4	4	2	2	2	4
(5,1)	0	4	4	2	2	2	4

Example

Here are the different cross sections of the workspace obtained for the sample points from the fiber above the cell $(5, 1)$. The drawings are from SCILAB.

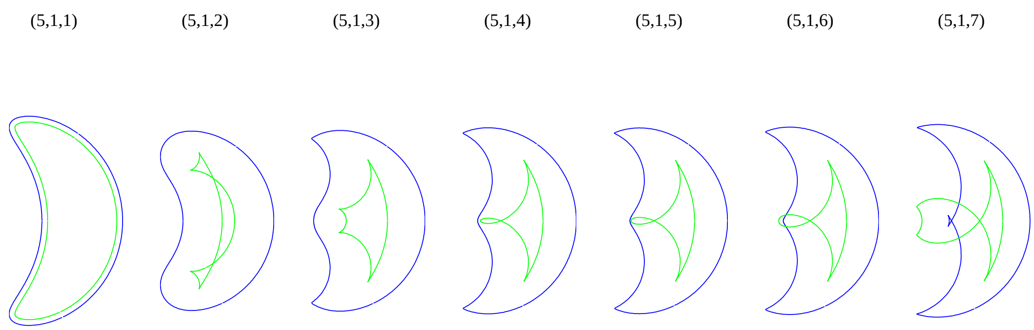

Figure 3: Fiber above the cell $(5, 1)$

We can verify on those drawings that the number of cusp points corresponds to the corresponding number in the precedent table.

6 Conclusion

In this paper, we manage to compute an algebraic set depending on the design parameters, such that in each connected component of its complementary the cuspidality behavior is the same . Thanks to an adapted CAD, we obtained a decomposition of this set in cells and at least one sample point,with rationnal coordinates, in the interior of each cell of higher dimension.

Among those sample points, P. Wenger's team found just one class of homotopy, in the torus representing the joint space, over the seven classes they encountered before for the cuspidal manipulators [10].

Generalization of this work to manipulators with one more design parameter (the parameter r_3 of the DH parameters) would permit to find representants of other classes. We are now working on.

Future research work is to try to generalize our study to other problems involving system of polynomial equations depending on parameters.

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